

Quantization of the shift of argument subalgebras in type A

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Abstract

Given a simple Lie algebra \mathfrak{g} and an element $\mu \in \mathfrak{g}^*$, the corresponding shift of argument subalgebra of $S(\mathfrak{g})$ is Poisson commutative. In the case where μ is regular, this subalgebra is known to admit a quantization, that is, it can be lifted to a commutative subalgebra of $U(\mathfrak{g})$. We show that if \mathfrak{g} is of type A , then this property extends to arbitrary μ , thus proving a conjecture of Feigin, Frenkel and Toledano Laredo. The proof relies on an explicit construction of generators of the center of the affine vertex algebra at the critical level.

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1 Introduction

Shift of argument subalgebras. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} with basis elements Y_1, \dots, Y_l and the corresponding structure constants c_{ij}^k . The symmetric algebra $S(\mathfrak{g})$ can be equipped with the *Lie–Poisson bracket* defined on the elements of the Lie algebra by

$$\{Y_i, Y_j\} = \sum_{k=1}^l c_{ij}^k Y_k. \quad (1.1)$$

Let $P = P(Y_1, \dots, Y_l)$ be an element of $S(\mathfrak{g})$ of a certain degree d . Fix any element $\mu \in \mathfrak{g}^*$ and let z be a variable. Make the substitution $Y_i \mapsto Y_i + z\mu(Y_i)$ and expand as a polynomial in z ,

$$P(Y_1 + z\mu(Y_1), \dots, Y_l + z\mu(Y_l)) = P^{(0)} + P^{(1)}z + \dots + P^{(d)}z^d$$

to define elements $P^{(i)} \in S(\mathfrak{g})$ associated with P and μ . Denote by $\overline{\mathcal{A}}_\mu$ the subalgebra of $S(\mathfrak{g})$ generated by all elements $P^{(i)}$ associated with all \mathfrak{g} -invariants $P \in S(\mathfrak{g})^{\mathfrak{g}}$. The subalgebra $\overline{\mathcal{A}}_\mu$ of $S(\mathfrak{g})$ is known as the *Mishchenko–Fomenko subalgebra* or *shift of argument subalgebra*. Its key property observed in [12] states that $\overline{\mathcal{A}}_\mu$ is Poisson commutative; that is, $\{R, S\} = 0$ for any elements $R, S \in \overline{\mathcal{A}}_\mu$.

We will identify \mathfrak{g}^* with \mathfrak{g} via a symmetric invariant bilinear form (see (1.2) below) and let n denote the rank of \mathfrak{g} . An element $\mu \in \mathfrak{g}^* \cong \mathfrak{g}$ is called *regular*, if the centralizer \mathfrak{g}^μ of μ in \mathfrak{g} has minimal possible dimension; this minimal dimension coincides with n . The subalgebra $S(\mathfrak{g})^{\mathfrak{g}}$ admits a family P_1, \dots, P_n of algebraically independent generators of respective degrees d_1, \dots, d_n . If the element $\mu \in \mathfrak{g}^*$ is regular, then $\overline{\mathcal{A}}_\mu$ has the properties:

- i) the subalgebra $\overline{\mathcal{A}}_\mu$ of $S(\mathfrak{g})$ is maximal Poisson commutative;
- ii) the elements $P_k^{(i)}$ with $k = 1, \dots, n$ and $i = 0, 1, \dots, d_k - 1$, are algebraically independent generators of $\overline{\mathcal{A}}_\mu$.

Property i) is a theorem of Panyushev and Yakimova [16]; the case of regular semisimple μ is due to Tarasov [20]. Property ii) is due to Bolsinov [1]; the regular semisimple case goes back to the original paper [12]. Another proof of ii) was given in [8].

Vinberg’s problem. The universal enveloping algebra $U(\mathfrak{g})$ is equipped with a canonical filtration and the associated graded algebra $\text{gr } U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$. Given that the subalgebra $\overline{\mathcal{A}}_\mu$ of $S(\mathfrak{g})$ is Poisson commutative, one could look for a commutative subalgebra \mathcal{A}_μ of $U(\mathfrak{g})$ which “quantizes” $\overline{\mathcal{A}}_\mu$ in the sense that $\text{gr } \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$. This quantization problem was raised by Vinberg in [21], where, in particular, some commuting families of elements of $U(\mathfrak{g})$ were produced. A positive solution of Vinberg’s problem was given by Rybnikov [17]

(for regular semisimple μ) and Feigin, Frenkel and Toledano Laredo [8] (for any regular μ) with the use of the center of the associated affine vertex algebra at the critical level (also known as the *Feigin–Frenkel center*). To briefly outline the solution, equip \mathfrak{g} with a standard symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ defined as the normalized Killing form

$$\langle X, Y \rangle = \frac{1}{2h^\vee} \text{tr} (\text{ad } X \text{ ad } Y), \quad (1.2)$$

where h^\vee is the *dual Coxeter number* for \mathfrak{g} . The corresponding *affine Kac–Moody algebra* $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \quad (1.3)$$

where $\mathfrak{g}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in \mathfrak{g} . For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we set $X[r] = X t^r$. The commutation relations of the Lie algebra $\widehat{\mathfrak{g}}$ have the form

$$[X[r], Y[s]] = [X, Y][r+s] + r \delta_{r,-s} \langle X, Y \rangle K, \quad X, Y \in \mathfrak{g},$$

and the element K is central in $\widehat{\mathfrak{g}}$. For any $\kappa \in \mathbb{C}$ denote by $U_\kappa(\widehat{\mathfrak{g}})$ the quotient of $U(\widehat{\mathfrak{g}})$ by the ideal generated by $K - \kappa$. The value $\kappa = -h^\vee$ corresponds to the *critical level*. Let I denote the left ideal of $U_{-h^\vee}(\widehat{\mathfrak{g}})$ generated by $\mathfrak{g}[t]$ and let $\text{Norm } I$ be its normalizer,

$$\text{Norm } I = \{v \in U_{-h^\vee}(\widehat{\mathfrak{g}}) \mid Iv \subseteq I\}.$$

The normalizer is a subalgebra of $U_{-h^\vee}(\widehat{\mathfrak{g}})$, and I is a two-sided ideal of $\text{Norm } I$. The *Feigin–Frenkel center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the associative algebra defined as the quotient

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \text{Norm } I / I. \quad (1.4)$$

By the Poincaré–Birkhoff–Witt theorem, the quotient of the algebra $U_{-h^\vee}(\widehat{\mathfrak{g}})$ by the left ideal I is isomorphic to the universal enveloping algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$, as a vector space. Hence, we have a vector space embedding

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow U(t^{-1}\mathfrak{g}[t^{-1}]).$$

Since $U(t^{-1}\mathfrak{g}[t^{-1}])$ is a subalgebra of $U_{-h^\vee}(\widehat{\mathfrak{g}})$, the embedding is an algebra homomorphism so that the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be regarded as a subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$. In fact, this subalgebra is *commutative* which is not immediate from the definition, but can be seen by identifying $\mathfrak{z}(\widehat{\mathfrak{g}})$ with the center of the affine vertex algebra at the critical level. Furthermore, by a theorem of Feigin and Frenkel [7] (see [10] for a detailed exposition), there exist elements $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$ such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \ r \geq 0], \quad (1.5)$$

where T is the derivation of the algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$ which is determined by the property that its commutator with the operator of left multiplication by $X[r]$ is found by

$$[T, X[r]] = -r X[r-1], \quad X \in \mathfrak{g}, \quad r < 0.$$

We will call such family S_1, \dots, S_n a *complete set of Segal–Sugawara vectors* for \mathfrak{g} . Another derivation D of the algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$ is determined by the property

$$[D, X[r]] = -r X[r], \quad X \in \mathfrak{g}, \quad r < 0;$$

and D defines a grading on $U(t^{-1}\mathfrak{g}[t^{-1}])$.

Given any element $\mu \in \mathfrak{g}^*$ and a nonzero $z \in \mathbb{C}$, the mapping

$$\varrho_{\mu,z} : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(\mathfrak{g}), \quad X[r] \mapsto Xz^r + \delta_{r,-1}\mu(X), \quad X \in \mathfrak{g}, \quad (1.6)$$

defines an algebra homomorphism. The image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\varrho_{\mu,z}$ is a commutative subalgebra of $U(\mathfrak{g})$. It does not depend on z and is denoted by \mathcal{A}_μ . If $S \in U(t^{-1}\mathfrak{g}[t^{-1}])$ is an element of degree d with respect to the grading defined by D , then regarding $\varrho_{\mu,z}(S)$ as a polynomial in z^{-1} , define the elements $S^{(i)} \in U(\mathfrak{g})$ by the expansion

$$\varrho_{\mu,z}(S) = S^{(0)}z^{-d} + \dots + S^{(d-1)}z^{-1} + S^{(d)}. \quad (1.7)$$

If $\mu \in \mathfrak{g}^*$ is regular then the following holds:

- i) the subalgebra \mathcal{A}_μ of $U(\mathfrak{g})$ is maximal commutative;
- ii) if $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$ are elements of the respective degrees d_1, \dots, d_n satisfying (1.5), then the elements $S_k^{(i)}$ with $k = 1, \dots, n$ and $i = 0, 1, \dots, d_k - 1$ are algebraically independent generators of \mathcal{A}_μ ;
- iii) $\text{gr } \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

This is derived with the use of the respective properties of the algebra $\overline{\mathcal{A}}_\mu$; see [8] for proofs. The subalgebra \mathcal{A}_μ was further studied in [9] where its spectra in finite-dimensional irreducible representations of \mathfrak{g} were described.

Note that both algebras \mathcal{A}_μ and $\overline{\mathcal{A}}_\mu$ are defined for arbitrary elements $\mu \in \mathfrak{g}^*$. Given that the property *iii)* holds for all regular μ , it was conjectured in [8, Conjecture 1], that this property is valid for all μ . As a consequence of our main result, we obtain a proof of this conjecture for type A ; see the Main Theorem below. In particular, this gives another proof of *iii)* for regular μ . More precisely, we will work with the reductive Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ and consider the respective subalgebras $\overline{\mathcal{A}}_\mu \subset S(\mathfrak{gl}_n)$ and $\mathcal{A}_\mu \subset U(\mathfrak{gl}_n)$. The proof will be based on the use of explicit formulas for generators of \mathcal{A}_μ .

Generators of \mathcal{A}_μ . For the Lie algebras \mathfrak{g} of type A , a few families of explicit generators S_1, \dots, S_n of $\mathfrak{z}(\widehat{\mathfrak{g}})$, and hence generators of the subalgebra \mathcal{A}_μ , were produced by Chervov and Talalaev [6] by extending Talalaev's work [18]; see also [4] and [14] where more direct proofs were given. In types B , C and D such explicit generators were constructed in [13]. Note also earlier work of Nazarov and Olshanski [15], where maximal commutative subalgebras of $U(\mathfrak{g})$ were produced with the use of Yangians; they quantize the Poisson algebras $\overline{\mathcal{A}}_\mu$ in all classical types for the case of regular semisimple μ . In a different form, a quantization of $\overline{\mathcal{A}}_\mu$ in type A was provided by Tarasov [19] via a symmetrization map.

We will work with a particular family of generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ which we recall below in Sec. 2. They allow us to define the associated family of generators $\phi_m^{(k)}$ with $m = 1, \dots, n$ and $k = 0, \dots, m-1$ of the subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{gl}_n)$; see (4.2) below. These generators are algebraically independent if μ is regular.

Our main result provides a way to choose an algebraically independent family of generators $\phi_m^{(k)}$ of \mathcal{A}_μ for an arbitrary element μ . To describe this subset, we will identify \mathfrak{gl}_n^* with \mathfrak{gl}_n via a symmetric bilinear form and regard μ as an $n \times n$ matrix. Suppose that the distinct eigenvalues of μ are $\lambda_1, \dots, \lambda_r$ and the Jordan canonical form of μ is the direct sum of the respective Jordan blocks $J_{\alpha_j^{(i)}}(\lambda_i)$ of sizes $\alpha_1^{(i)} \geq \alpha_2^{(i)} \geq \dots \geq \alpha_{s_i}^{(i)} \geq 1$. We let $\alpha^{(i)}$ denote the corresponding Young diagram whose j -th row is $\alpha_j^{(i)}$ and let $|\alpha^{(i)}|$ be the number of boxes of $\alpha^{(i)}$. Given these data, introduce another Young diagram $\gamma = (\gamma_1, \gamma_2, \dots)$ by setting

$$\gamma_l = \sum_{i=1}^r \sum_{j \geq l+1} \alpha_j^{(i)}, \quad (1.8)$$

so that γ_l is the total number of boxes which are strictly below the l -th rows in all diagrams $\alpha^{(i)}$. Furthermore, associate the elements of the family $\phi_m^{(k)}$ with boxes of the diagram $\Gamma = (n, n-1, \dots, 1)$ so that the (i, j) box of Γ corresponds to $\phi_{n-j+1}^{(n-i-j+1)}$, as illustrated:

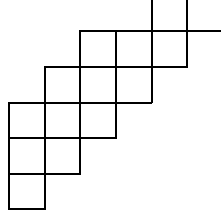
$$\Gamma = \begin{array}{cccccc} \phi_n^{(n-1)} & \phi_{n-1}^{(n-2)} & \dots & \phi_2^{(1)} & \phi_1^{(0)} \\ \phi_n^{(n-2)} & \phi_{n-1}^{(n-3)} & \dots & \phi_2^{(0)} & \\ \dots & \dots & \dots & & \\ \phi_n^{(1)} & \phi_{n-1}^{(0)} & & & \\ \phi_n^{(0)} & & & & \end{array} \quad (1.9)$$

Note that the diagram γ is contained in Γ . We can now state our main theorem, where μ is an arbitrary element of \mathfrak{gl}_n and Γ/γ is the associated skew diagram.

Main Theorem. *The elements $\phi_m^{(k)}$ corresponding to the boxes of the skew diagram Γ/γ are algebraically independent generators of the subalgebra \mathcal{A}_μ . Moreover, the subalgebra \mathcal{A}_μ is a quantization of $\overline{\mathcal{A}}_\mu$ so that $\text{gr } \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.*

By considering some other complete sets of Segal–Sugawara vectors, we also show that the first part of the Main Theorem remains valid if the elements $\phi_m^{(k)}$ are replaced with those of other families; see Corollaries 4.5 and 4.6 below.

Example 1.1. Take $n = 6$ and let μ be a nilpotent matrix with the Jordan blocks of sizes $(2, 2, 1, 1)$. Then $\gamma = (4, 2, 1)$ and the skew diagram Γ/γ is



so that the algebraically independent generators of \mathcal{A}_μ are those corresponding to the boxes of Γ , excluding $\phi_3^{(2)}$, $\phi_4^{(3)}$, $\phi_5^{(3)}$, $\phi_5^{(4)}$, $\phi_6^{(3)}$, $\phi_6^{(4)}$ and $\phi_6^{(5)}$. \square

Note also two extreme cases. If μ is regular, then all Jordan blocks correspond to distinct eigenvalues so that each $\alpha^{(i)}$ is a single row diagram. Therefore, $\gamma = \emptyset$, so that all generators $\phi_m^{(k)}$ associated with the boxes of Γ are algebraically independent. On the other hand, for scalar matrices μ we have $\gamma = (n - 1, n - 2, \dots, 1)$. In this case, \mathcal{A}_μ is generated by $\phi_1^{(0)}, \dots, \phi_n^{(0)}$ and it coincides with the center of $U(\mathfrak{gl}_n)$.

Our proofs rely on *Bolsinov's completeness criterion* [1, Theorem 3.2] which applies to the shift of argument subalgebras associated with an arbitrary Lie algebra \mathfrak{g} . The required condition for reductive Lie algebras is the equality

$$\text{ind } \mathfrak{g} = \text{ind } \mathfrak{g}^\mu \quad (1.10)$$

of the indices of \mathfrak{g} and the centralizer \mathfrak{g}^μ of μ in \mathfrak{g} , where the *index* of an arbitrary Lie algebra \mathfrak{g} is the minimal dimension of the stabilizers \mathfrak{g}^x , $x \in \mathfrak{g}^*$, for the coadjoint representation. In the case $\mathfrak{g} = \mathfrak{gl}_n$ and arbitrary $\mu \in \mathfrak{g}$ this equality was claimed to be verified by Bolsinov [2, Sec. 3] (and was suggested to be extendable to arbitrary semisimple Lie algebras) and by Elashvili (private communication), but details were not published. The first published proof is due to Yakimova [22], which extends to all classical Lie algebras. The equality (1.10) is widely referred to as the *Elashvili conjecture*, but should rather be called the *Bolsinov–Elashvili conjecture*¹; see e.g. [3] for its proof covering all simple Lie algebras and more references.

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¹ A. Elashvili kindly informed us that the conjectural equality had emerged from A. Bolsinov's questions to him and so it should also be attributed to the author of [1].

2 Generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$

For $i, j \in \{1, \dots, n\}$ we will denote by E_{ij} the standard basis elements of \mathfrak{gl}_n . We extend the form (1.2) to the invariant symmetric bilinear form on \mathfrak{gl}_n which is given by

$$\langle X, Y \rangle = \text{tr}(XY) - \frac{1}{n} \text{tr} X \text{tr} Y, \quad X, Y \in \mathfrak{gl}_n,$$

where X and Y are regarded as $n \times n$ matrices. Note that the kernel of the form is spanned by the element $E_{11} + \dots + E_{nn}$, and its restriction to the subalgebra \mathfrak{sl}_n is given by

$$\langle X, Y \rangle = \text{tr}(XY), \quad X, Y \in \mathfrak{sl}_n.$$

The affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n[t, t^{-1}] \oplus \mathbb{C}K$ has the commutation relations

$$[E_{ij}[r], E_{kl}[s]] = \delta_{kj} E_{il}[r+s] - \delta_{il} E_{kj}[r+s] + r \delta_{r,-s} K \left(\delta_{kj} \delta_{il} - \frac{\delta_{ij} \delta_{kl}}{n} \right), \quad (2.1)$$

and the element K is central. The critical level $-n$ coincides with the negative of the dual Coxeter number for \mathfrak{sl}_n . We will work with the extended Lie algebra $\widehat{\mathfrak{gl}}_n \oplus \mathbb{C}\tau$ where the additional element τ satisfies the commutation relations

$$[\tau, X[r]] = -r X[r-1], \quad [\tau, K] = 0. \quad (2.2)$$

For any $r \in \mathbb{Z}$ combine the elements $E_{ij}[r]$ into the matrix $E[r]$ so that

$$E[r] = \sum_{i,j=1}^n e_{ij} \otimes E_{ij}[r] \in \text{End } \mathbb{C}^n \otimes \mathbb{U}, \quad (2.3)$$

where the e_{ij} are the standard matrix units and \mathbb{U} stands for the universal enveloping algebra of $\widehat{\mathfrak{gl}}_n \oplus \mathbb{C}\tau$. For each $a \in \{1, \dots, m\}$ introduce the element $E[r]_a$ of the algebra

$$\underbrace{\text{End } \mathbb{C}^n \otimes \dots \otimes \text{End } \mathbb{C}^n}_m \otimes \mathbb{U} \quad (2.4)$$

by

$$E[r]_a = \sum_{i,j=1}^n 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes E_{ij}[r]. \quad (2.5)$$

We let $H^{(m)}$ and $A^{(m)}$ denote the respective images of the symmetrizer $h^{(m)}$ and anti-symmetrizer $a^{(m)}$ in the group algebra for the symmetric group \mathfrak{S}_m under its natural action on $(\mathbb{C}^n)^{\otimes m}$. The elements $h^{(m)}$ and $a^{(m)}$ are the idempotents in the group algebra $\mathbb{C}[\mathfrak{S}_m]$ defined by

$$h^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} s \quad \text{and} \quad a^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \text{sgn } s \cdot s.$$

We will identify $H^{(m)}$ and $A^{(m)}$ with the respective elements $H^{(m)} \otimes 1$ and $A^{(m)} \otimes 1$ of the algebra (2.4). Define the elements $\phi_{ma}, \psi_{ma}, \theta_{ma} \in U(t^{-1}\mathfrak{gl}_n[t^{-1}])$ by the expansions

$$\mathrm{tr}_{1,\dots,m} A^{(m)}(\tau + E[-1]_1) \dots (\tau + E[-1]_m) = \phi_{m0} \tau^m + \phi_{m1} \tau^{m-1} + \dots + \phi_{mm}, \quad (2.6)$$

$$\mathrm{tr}_{1,\dots,m} H^{(m)}(\tau + E[-1]_1) \dots (\tau + E[-1]_m) = \psi_{m0} \tau^m + \psi_{m1} \tau^{m-1} + \dots + \psi_{mm}, \quad (2.7)$$

where the traces are taken with respect to all m copies of $\mathrm{End} \mathbb{C}^n$ in (2.4), and

$$\mathrm{tr}(\tau + E[-1])^m = \theta_{m0} \tau^m + \theta_{m1} \tau^{m-1} + \dots + \theta_{mm}. \quad (2.8)$$

Expressions like $\tau + E[-1]$ are understood as matrices, where τ is regarded as the scalar matrix $\tau 1$. Furthermore, introduce the *column-determinant* of the matrix $\tau + E[-1]$ by

$$\mathrm{cdet}(\tau + E[-1]) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn} \sigma \cdot (\tau + E[-1])_{\sigma(1)1} \dots (\tau + E[-1])_{\sigma(n)n} \quad (2.9)$$

and expand it as a polynomial in τ ,

$$\mathrm{cdet}(\tau + E[-1]) = \tau^n + \phi_1 \tau^{n-1} + \dots + \phi_n, \quad \phi_m \in U(t^{-1}\mathfrak{gl}_n[t^{-1}]). \quad (2.10)$$

We have the expansion of the noncommutative characteristic polynomial,

$$\mathrm{cdet}(u + \tau + E[-1]) = \sum_{m=0}^n u^{n-m} \mathrm{tr}_{1,\dots,m} A^{(m)}(\tau + E[-1]_1) \dots (\tau + E[-1]_m), \quad (2.11)$$

where u is a variable. This implies the relations

$$\phi_{ma} = \binom{n-a}{m-a} \phi_a, \quad 0 \leq a \leq m \leq n. \quad (2.12)$$

In particular, $\phi_{mm} = \phi_m$ for $m = 1, \dots, n$.

Theorem 2.1. *All elements ϕ_m , ψ_{ma} and θ_{ma} belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$. Moreover, each of the families*

$$\phi_1, \dots, \phi_n, \quad \psi_{11}, \dots, \psi_{nn} \quad \text{and} \quad \theta_{11}, \dots, \theta_{nn}$$

is a complete set of Segal–Sugawara vectors for \mathfrak{gl}_n . \square

This theorem goes back to [6], where the elements ϕ_m were first discovered (in a slightly different form). A direct proof of the theorem was given in [4]. The elements ψ_{ma} are related to ϕ_{ma} through the quantum MacMahon Master Theorem of [11], while a relationship between the ϕ_{ma} and θ_{ma} is provided by a Newton-type identity given in [5, Theorem 15]. Note that super-versions of these relations between the families of Segal–Sugawara vectors for the Lie superalgebra $\mathfrak{gl}_{m|n}$ were given in the paper [14], which also provides simpler arguments in the purely even case.

3 Generators of \mathcal{A}_μ

In accordance with the results which we recalled in the Introduction, the application of the homomorphism (1.6) to elements of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ provided by Theorem 2.1 yields the corresponding families of elements of the subalgebra $\mathcal{A}_\mu \subset U(\mathfrak{gl}_n)$ through the expansion (1.7). To give explicit formulas, we will use the tensor product algebra (2.4), where U will now denote the algebra of differential operators whose elements are finite sums of the form

$$\sum_{k,l \geq 0} u_{kl} z^{-k} \partial_z^l, \quad u_{kl} \in U(\mathfrak{gl}_n).$$

Note that ∂_z emerges here as the image of the element $-\tau$ under the extension of the homomorphism (1.6). As in (2.3), we set

$$E = \sum_{i,j=1}^n e_{ij} \otimes E_{ij} \in \text{End } \mathbb{C}^n \otimes U(\mathfrak{gl}_n),$$

and extend the notation (2.5) to the matrices E , μ and $M = -\partial_z + \mu + Ez^{-1}$. Assuming that $\mu \in \mathfrak{gl}_n$ is arbitrary, introduce the polynomials $\phi_{ma}(z)$, $\psi_{ma}(z)$ and $\theta_{ma}(z)$ in z^{-1} (depending on μ) with coefficients in $U(\mathfrak{gl}_n)$ by the expansions

$$\begin{aligned} \text{tr}_{1,\dots,m} A^{(m)} M_1 \dots M_m &= \phi_{m0}(z) \partial_z^m + \phi_{m1}(z) \partial_z^{m-1} + \dots + \phi_{mm}(z), \\ \text{tr}_{1,\dots,m} H^{(m)} M_1 \dots M_m &= \psi_{m0}(z) \partial_z^m + \psi_{m1}(z) \partial_z^{m-1} + \dots + \psi_{mm}(z), \end{aligned}$$

and

$$\text{tr } M^m = \theta_{m0}(z) \partial_z^m + \theta_{m1}(z) \partial_z^{m-1} + \dots + \theta_{mm}(z).$$

Furthermore, following (2.10) define the polynomials $\phi_a(z)$ by expanding the column-determinant

$$\text{cdet } M = \phi_0(z) \partial_z^n + \phi_1(z) \partial_z^{n-1} + \dots + \phi_n(z). \quad (3.1)$$

By (2.12) we have

$$\phi_{ma}(z) = \binom{n-a}{m-a} \phi_a(z), \quad 0 \leq a \leq m \leq n,$$

and so $\phi_{mm}(z) = \phi_m(z)$ for all m . Introduce the coefficients of polynomials by

$$\begin{aligned} \phi_m(z) &= \phi_m^{(0)} z^{-m} + \dots + \phi_m^{(m-1)} z^{-1} + \phi_m^{(m)}, \\ \psi_{mm}(z) &= \psi_{mm}^{(0)} z^{-m} + \dots + \psi_{mm}^{(m-1)} z^{-1} + \psi_{mm}^{(m)}, \end{aligned}$$

and

$$\theta_{mm}(z) = \theta_{mm}^{(0)} z^{-m} + \dots + \theta_{mm}^{(m-1)} z^{-1} + \theta_{mm}^{(m)}.$$

By Theorem 2.1 and the general results of [8] and [17] we get the following.

Theorem 3.1. *Given any $\mu \in \mathfrak{gl}_n$, all coefficients of the polynomials $\phi_m(z)$, $\psi_{ma}(z)$ and $\theta_{ma}(z)$ belong to the commutative subalgebra \mathcal{A}_μ of $U(\mathfrak{gl}_n)$. Moreover, the elements of each of the families*

$$\phi_m^{(k)}, \quad \psi_{mm}^{(k)} \quad \text{and} \quad \theta_{mm}^{(k)}$$

with $m = 1, \dots, n$ and $k = 0, 1, \dots, m-1$, are generators of the algebra \mathcal{A}_μ . If μ is regular, then each of these families is algebraically independent. \square

Example 3.2. Using the family $\theta_{mm}^{(k)}$ we get the following algebraically independent generators of the algebra \mathcal{A}_μ for regular μ :

$$\begin{aligned} \text{for } \mathfrak{gl}_2 : & \quad \text{tr } E, \quad \text{tr } \mu E, \quad \text{tr } E^2 \\ \text{for } \mathfrak{gl}_3 : & \quad \text{tr } E, \quad \text{tr } \mu E, \quad \text{tr } \mu^2 E, \quad \text{tr } E^2, \quad \text{tr } \mu E^2, \quad \text{tr } E^3 \\ \text{for } \mathfrak{gl}_4 : & \quad \text{tr } E, \quad \text{tr } \mu E, \quad \text{tr } \mu^2 E, \quad \text{tr } \mu^3 E, \quad \text{tr } E^2, \quad \text{tr } \mu E^2, \\ & \quad 2 \text{tr } \mu^2 E^2 + \text{tr } (\mu E)^2, \quad \text{tr } E^3, \quad \text{tr } \mu E^3, \quad \text{tr } E^4. \end{aligned}$$

4 Proof of the Main Theorem

Note that $M = -\partial_z + \mu + Ez^{-1}$ is a *Manin matrix* and therefore the polynomials $\phi_{ma}(z)$ and $\psi_{ma}(z)$ admit expressions in terms of noncommutative minors and permanents. In more detail, given two subsets $B = \{b_1, \dots, b_k\}$ and $C = \{c_1, \dots, c_k\}$ of $\{1, \dots, n\}$ we will consider the corresponding column-minor

$$M_C^B = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn } \sigma \cdot M_{b_{\sigma(1)}c_1} \dots M_{b_{\sigma(k)}c_k}.$$

By [5, Proposition 18] (see also [14, Proposition 2.1]) we have

$$A^{(m)} M_1 \dots M_m = A^{(m)} M_1 \dots M_m A^{(m)},$$

which implies

$$\text{tr}_{1, \dots, m} A^{(m)} M_1 \dots M_m = \sum_{I, |I|=m} M_I^I, \quad (4.1)$$

summed over the subsets $I = \{i_1, \dots, i_m\}$ with $i_1 < \dots < i_m$. By Theorem 3.1, the algebra \mathcal{A}_μ is generated by the coefficients $\phi_m^{(k)}$ of the constant term of the differential operator,

$$\phi_m^{(0)} z^{-m} + \dots + \phi_m^{(m-1)} z^{-1} + \phi_m^{(m)} = \sum_{I, |I|=m} M_I^I 1,$$

assuming that $\partial_z 1 = 0$. This implies the formula

$$\phi_m^{(k)} = z^{m-k} \sum_{I, |I|=m} \sum_{\substack{B, C \subset I \\ |B|=|C|=k}} \text{sgn } \sigma \cdot \mu_C^B [-\partial_z + Ez^{-1}]_{I \setminus C}^I 1, \quad (4.2)$$

where σ denotes the permutation of the set I given by

$$\sigma = \begin{pmatrix} B, & I \setminus B \\ C, & I \setminus C \end{pmatrix} = \begin{pmatrix} b_1, \dots, b_k, i_1, \dots, \widehat{b}_1, \dots, \widehat{b}_k, \dots, i_m \\ c_1, \dots, c_k, i_1, \dots, \widehat{c}_1, \dots, \widehat{c}_k, \dots, i_m \end{pmatrix},$$

and we assume that $b_1 < \dots < b_k$ and $c_1 < \dots < c_k$ for the respective elements of the subsets B and C in I .

For each $l = 1, \dots, n$ introduce the polynomial in a variable t with coefficients in \mathcal{A}_μ by

$$\Phi_l(t, \mu) = \phi_l^{(0)}(\mu) t^{n-l} + \phi_{l+1}^{(1)}(\mu) t^{n-l-1} + \dots + \phi_n^{(n-l)}(\mu), \quad (4.3)$$

where the elements $\phi_m^{(k)} = \phi_m^{(k)}(\mu)$ are defined in (4.2) and we indicated dependence of μ . The coefficients of $\Phi_l(t, \mu)$ are the elements of the l -th row of the diagram Γ ; see (1.9).

Lemma 4.1. *For any $a \in \mathbb{C}$ we have the relation*

$$\Phi_l(t, \mu + a1) = \Phi_l(t + a, \mu).$$

Proof. We have

$$\mathrm{tr}_{1, \dots, m} A^{(m)}(a + M_1) \dots (a + M_m) = \sum_{p=0}^m a^p \sum_{i_1 < \dots < i_{m-p}} \mathrm{tr}_{1, \dots, m} A^{(m)} M_{i_1} \dots M_{i_{m-p}}.$$

Furthermore, $A^{(m)} = \mathrm{sgn} p \cdot A^{(m)} P$ for any $p \in \mathfrak{S}_m$, where P denotes the image of p in the algebra (2.4) under the action of \mathfrak{S}_m . Hence, applying conjugations by appropriate elements P and using the cyclic property of trace, we can write the expression as

$$\sum_{p=0}^m \binom{m}{p} a^p \mathrm{tr}_{1, \dots, m} A^{(m)} M_1 \dots M_{m-p}.$$

The partial trace of the anti-symmetrizer over the m -th copy of $\mathrm{End} \mathbb{C}^n$ is found by

$$\mathrm{tr}_m A^{(m)} = \frac{n - m + 1}{m} A^{(m-1)} \quad (4.4)$$

which implies

$$\mathrm{tr}_{m-p+1, \dots, m} A^{(m)} = \frac{(n - m + p)! (m - p)!}{(n - m)! m!} A^{(m-p)}.$$

Hence,

$$\mathrm{tr}_{1, \dots, m} A^{(m)}(a + M_1) \dots (a + M_m) = \sum_{p=0}^m \binom{n - m + p}{p} a^p \mathrm{tr}_{1, \dots, m-p} A^{(m-p)} M_1 \dots M_{m-p}.$$

Now equate the constant terms of the differential operators on both sides and take the coefficients of z^{-m+k} to get the relation

$$\phi_m^{(k)}(\mu + a1) = \sum_{p=0}^k \binom{n-m+p}{p} a^p \phi_{m-p}^{(k-p)}(\mu).$$

Therefore, for the polynomial $\Phi_l(t, \mu + a1)$ we find

$$\begin{aligned} \Phi_l(t, \mu + a1) &= \sum_{k=0}^{n-l} \phi_{l+k}^{(k)}(\mu + a1) t^{n-l-k} = \sum_{k=0}^{n-l} \sum_{p=0}^k \binom{n-l-k+p}{p} a^p \phi_{l+k-p}^{(k-p)}(\mu) t^{n-l-k} \\ &= \sum_{p=0}^{n-l} a^p \sum_{r=0}^{n-l-p} \binom{n-l-r}{p} \phi_{l+r}^{(r)}(\mu) t^{n-l-p-r}, \end{aligned}$$

which coincides with

$$\sum_{p=0}^{n-l} \frac{a^p}{p!} \left(\frac{d}{dt} \right)^p \Phi_l(t, \mu) = \Phi_l(t + a, \mu),$$

as claimed. \square

Lemma 4.2. *Suppose that μ has the form of a block-diagonal matrix*

$$\mu = \begin{bmatrix} J_\alpha(0) & \mathbf{O} \\ \mathbf{O} & \tilde{\mu} \end{bmatrix}, \quad (4.5)$$

where $J_\alpha(0)$ is the nilpotent Jordan matrix associated with a diagram $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\tilde{\mu}$ is an arbitrary square matrix of size q such that $|\alpha| + q = n$. Then for any $l \geq 1$ we have

$$\phi_{l+k}^{(k)} = 0 \quad \text{for all } n - l - \delta_l + 1 \leq k \leq n - l,$$

where $\delta_l = \alpha_{l+1} + \alpha_{l+2} + \dots$ is the number of boxes of α below its row l .

Proof. The generator $\phi_{l+k}^{(k)}$ is found by (4.2) for $m = l + k$. The internal sum is a linear combination of $k \times k$ minors of the matrix μ satisfying the condition that the union $B \cup C$ of the row and column indices of each minor is a set of size not exceeding $k + l$. On the other hand, with the given condition on k , the minor μ_C^B can be nonzero only if the union of row and column indices is of the size at least $k + l + 1$. Indeed, this follows from the observation that if p is a positive integer, then any nonzero $p \times p$ minor of a nilpotent Jordan block has the property that the minimal possible size of the union of its row and column indices is $p + 1$. However, the condition $k \geq n - l - \delta_l + 1$ means that $k \geq \alpha_1 + \dots + \alpha_l - l + 1 + q$. Therefore, a nonzero $k \times k$ minor must involve at least $l + 1$ Jordan blocks. \square

In the following we use the notation of the Main Theorem. In addition, for each diagram $\alpha^{(i)}$ we denote by $\delta_l^{(i)}$ the corresponding parameter δ_l , as defined in Lemma 4.2, so that for the number γ_l defined in (1.8) we have

$$\gamma_l = \sum_{i=1}^r \delta_l^{(i)}.$$

Corollary 4.3. *The polynomial $\Phi_l(t, \mu)$ admits the factorization*

$$\Phi_l(t, \mu) = (t + \lambda_1)^{\delta_l^{(1)}} \dots (t + \lambda_r)^{\delta_l^{(r)}} \tilde{\Phi}_l(t, \mu)$$

for a certain polynomial $\tilde{\Phi}_l(t, \mu)$ in t .

Proof. The algebra \mathcal{A}_μ is known to depend only on the adjoint orbit of μ ; see [8]. More precisely, as we can see from formulas (4.1), the elements $\phi_m^{(k)}$ are unchanged under the simultaneous replacements $\mu \mapsto g\mu g^{-1}$ and $E \mapsto g E g^{-1}$ for $g \in \mathrm{GL}_n$. This implies that $\mathcal{A}_{g\mu g^{-1}}$ can be identified with the algebra \mathcal{A}_μ associated with the image of $\mathrm{U}(\mathfrak{gl}_n)$ under the automorphism sending E to $g E g^{-1}$.

For any $i \in \{1, \dots, r\}$ the Jordan canonical form of $\mu - \lambda_i 1$ is a matrix of the form (4.5), where $\alpha = \alpha^{(i)}$. By Lemma 4.2, the polynomial $\Phi_l(t, \mu - \lambda_i 1)$ is divisible by $t^{\delta_l^{(i)}}$. Hence, by Lemma 4.1, the polynomial $\Phi_l(t, \mu) = \Phi_l(t + \lambda_i, \mu - \lambda_i 1)$ is divisible by $(t + \lambda_i)^{\delta_l^{(i)}}$. \square

We can now complete the proof of the Main Theorem. First, Corollary 4.3 implies that for any $l = 1, \dots, n$ the generators $\phi_{l+k}^{(k)}$ with $n - l - \gamma_l + 1 \leq k \leq n - l$ are linear combinations of those generators with $k = 0, 1, \dots, n - l - \gamma_l$. Therefore, the elements $\phi_{l+k}^{(k)}$ corresponding to the boxes of the skew diagram Γ/γ generate the algebra \mathcal{A}_μ . It remains to verify that these generators are algebraically independent.

Consider the elements $\bar{\phi}_m^{(k)} \in \mathrm{S}(\mathfrak{gl}_n)$ which are defined by

$$\bar{\phi}_m^{(k)} = \sum_{I, |I|=m} \sum_{\substack{B, C \subset I \\ |B|=|C|=k}} \mathrm{sgn} \sigma \cdot \mu_C^B E_{I \setminus C}^{I \setminus B}, \quad (4.6)$$

with the notation as in (4.2), where the entries of the matrix E are now regarded as elements of the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$. Equivalently, the elements $\bar{\phi}_m^{(k)}$ are found by

$$\mathrm{tr}_{1, \dots, m} A^{(m)}(\mu_1 + E_1 z^{-1}) \dots (\mu_m + E_m z^{-1}) = \bar{\phi}_m^{(0)} z^{-m} + \dots + \bar{\phi}_m^{(m-1)} z^{-1} + \bar{\phi}_m^{(m)}. \quad (4.7)$$

They are generators of the subalgebra $\bar{\mathcal{A}}_\mu$. The arguments of this section (including Lemmas 4.1, 4.2 and Corollary 4.3) applied to these generators instead of the $\phi_m^{(k)}$ show that the elements $\bar{\phi}_m^{(k)}$ corresponding to the boxes of the skew diagram Γ/γ generate the algebra $\bar{\mathcal{A}}_\mu$. Furthermore, we have the following.

Lemma 4.4. *The generators $\bar{\phi}_m^{(k)}$ of the subalgebra $\bar{\mathcal{A}}_\mu$ corresponding to the boxes of the skew diagram Γ/γ are algebraically independent.*

Proof. Regarding the elements $\bar{\phi}_m^{(k)}$ as polynomials in the variables E_{ij} , we will see that their differentials $d\bar{\phi}_m^{(k)}$ are linearly independent at a certain point. Since these elements generate $\bar{\mathcal{A}}_\mu$, the linear span of the differentials $d\bar{\phi}_m^{(k)}$ at any point coincides with the linear span of all differentials

$$d\bar{\mathcal{A}}_\mu = \text{span of } \{d\phi \mid \phi \in \bar{\mathcal{A}}_\mu\}.$$

On the other hand, Bolsinov's criterion [1, Theorem 3.2] implies that the relation

$$\dim d\bar{\mathcal{A}}_\mu = \text{rank } \mathfrak{gl}_n + \frac{1}{2}(\dim \mathfrak{gl}_n - \dim \mathfrak{gl}_n^\mu)$$

holds at a certain regular point if and only if the equality (1.10) holds for $\mathfrak{g} = \mathfrak{gl}_n$; see also [3, Theorem 2.7] for a concise exposition of this result. This equality does hold [22], and so, to show that the differentials $d\bar{\phi}_m^{(k)}$ of the generators are linearly independent at a certain point, we only need to verify that the number of boxes of the skew diagram Γ/γ coincides with

$$\text{rank } \mathfrak{gl}_n + \frac{1}{2}(\dim \mathfrak{gl}_n - \dim \mathfrak{gl}_n^\mu) = n + \frac{1}{2}(n^2 - \dim \mathfrak{gl}_n^\mu).$$

Since $|\Gamma| = n(n+1)/2$, the desired formula is equivalent to the relation

$$\dim \mathfrak{gl}_n^\mu = 2|\gamma| + n. \tag{4.8}$$

For the dimension of the centralizer we have

$$\dim \mathfrak{gl}_n^\mu = \sum_{i=1}^r \dim \mathfrak{gl}_{n_i}^{\mu^{(i)}},$$

where $\mu^{(i)}$ denotes the direct sum of all Jordan blocks of μ with the eigenvalue λ_i , and n_i is the size of $\mu^{(i)}$. Hence, by the definition of γ , the verification of (4.8) reduces to the case where μ has only one eigenvalue. Let $\alpha_1 \geq \dots \geq \alpha_s$ be the respective sizes of the Jordan blocks of such matrix μ . Then $\dim \mathfrak{gl}_n^\mu = \alpha_1 + 3\alpha_2 + \dots + (2s-1)\alpha_s$, while

$$|\gamma| = \alpha_2 + 2\alpha_3 + \dots + (s-1)\alpha_s \quad \text{and} \quad n = \alpha_1 + \dots + \alpha_s,$$

thus implying (4.8). \square

Now consider the generators $\phi_m^{(k)}$ of the algebra \mathcal{A}_μ associated with the boxes of the diagram Γ/γ . By Lemma 4.4, the corresponding elements $\bar{\phi}_m^{(k)}$ are nonzero, so that the image of $\phi_m^{(k)}$ in the $(m-k)$ -th component of $\text{gr } U(\mathfrak{gl}_n) \cong S(\mathfrak{gl}_n)$ coincides with $\bar{\phi}_m^{(k)}$. Moreover, the generators $\phi_m^{(k)}$ corresponding to the boxes of the diagram Γ/γ are algebraically

independent. This completes the proof of the first part of the Main Theorem, and the second part also follows.

Finally, we will extend the first part of the Main Theorem by providing some other families of algebraically independent generators of the algebra \mathcal{A}_μ . To this end, introduce the families $\overline{\psi}_m^{(k)}$ and $\overline{\theta}_m^{(k)}$ of generators of the algebra $\overline{\mathcal{A}}_\mu$ by the respective expansions

$$\mathrm{tr}_{1,\dots,m} H^{(m)}(\mu_1 + E_1 z^{-1}) \dots (\mu_m + E_m z^{-1}) = \overline{\psi}_m^{(0)} z^{-m} + \dots + \overline{\psi}_m^{(m-1)} z^{-1} + \overline{\psi}_m^{(m)} \quad (4.9)$$

and

$$\mathrm{tr}(\mu + E z^{-1})^m = \overline{\theta}_m^{(0)} z^{-m} + \dots + \overline{\theta}_m^{(m-1)} z^{-1} + \overline{\theta}_m^{(m)}, \quad (4.10)$$

where

$$E = \sum_{i,j=1}^n e_{ij} \otimes E_{ij} \in \mathrm{End} \mathbb{C}^n \otimes S(\mathfrak{gl}_n),$$

and extend the notation (2.5) to matrices E and μ . The polynomials $\overline{\phi}_m(z)$, $\overline{\psi}_m(z)$ and $\overline{\theta}_m(z)$ in z^{-1} given by the respective expressions in (4.7), (4.9) and (4.10) are related by the classical MacMahon Master Theorem and Newton's identities:

$$\sum_{l=0}^m (-1)^l \overline{\phi}_l(z) \overline{\psi}_{m-l}(z) = 0 \quad (4.11)$$

and

$$m \overline{\phi}_m(z) = \sum_{l=1}^m (-1)^{l-1} \overline{\theta}_l(z) \overline{\phi}_{m-l}(z) \quad (4.12)$$

for $m \geq 1$. Writing the relations (4.11) and (4.12) in terms of the coefficients of the polynomials, we find that each of the generators $\overline{\psi}_m^{(k)}$ and $\overline{\theta}_m^{(k)}$ with $m = 1, \dots, n$ and $k = 0, 1, \dots, m-1$ will be presented in the form

$$c \cdot \overline{\phi}_m^{(k)} + \text{linear combination of } \overline{\phi}_{m_1}^{(k_1)} \dots \overline{\phi}_{m_s}^{(k_s)}, \quad s \geq 2,$$

for a nonzero constant c , where $m_1 + \dots + m_s = m$ and $k_1 + \dots + k_s = k$. As we pointed out above, the elements $\overline{\phi}_m^{(k)}$ corresponding to a certain row of the diagram Γ are linear combinations of the elements of this row in the skew diagram Γ/γ . This implies that each of the families of generators $\overline{\psi}_m^{(k)}$ and $\overline{\theta}_m^{(k)}$ associated with the boxes of Γ/γ as in (1.9), is algebraically independent. This leads to the following corollary, where, as before, $\mu \in \mathfrak{gl}_n$ is an arbitrary matrix.

Corollary 4.5. *The elements of each of the two families $\psi_{mm}^{(k)}$ and $\theta_{mm}^{(k)}$ associated with the boxes of the skew diagram Γ/γ as in (1.9) are algebraically independent generators of the algebra \mathcal{A}_μ . \square*

To construct two more families of generators of the algebra \mathcal{A}_μ , define the elements $\varphi_m^{(k)}, \psi_m^{(k)} \in U(\mathfrak{gl}_n)$ by the expansions

$$\begin{aligned}\mathrm{tr}_{1,\dots,m} A^{(m)}(\mu_1 + E_1 z^{-1}) \dots (\mu_m + E_m z^{-1}) &= \varphi_m^{(0)} z^{-m} + \dots + \varphi_m^{(m-1)} z^{-1} + \varphi_m^{(m)}, \\ \mathrm{tr}_{1,\dots,m} H^{(m)}(\mu_1 + E_1 z^{-1}) \dots (\mu_m + E_m z^{-1}) &= \psi_m^{(0)} z^{-m} + \dots + \psi_m^{(m-1)} z^{-1} + \psi_m^{(m)}.\end{aligned}$$

It is easy to verify that each of the families $\varphi_m^{(k)}$ and $\psi_m^{(k)}$ with $m = 1, \dots, n$ and $k = 0, \dots, m-1$ generates the algebra \mathcal{A}_μ . Indeed, by Theorem 3.1, the algebra \mathcal{A}_μ is generated by the coefficients $\phi_m^{(k)}$ of the constant term of the differential operator,

$$\begin{aligned}\mathrm{tr}_{1,\dots,m} A^{(m)}(-\partial_z + \mu_1 + E_1 z^{-1}) \dots (-\partial_z + \mu_m + E_m z^{-1}) 1 \\ = \phi_m^{(0)} z^{-m} + \dots + \phi_m^{(m-1)} z^{-1} + \phi_m^{(m)}.\end{aligned}$$

Hence, $\phi_m^{(k)}$ is found as the coefficient of z^{-m+k} in the expression

$$\sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_{m-k}} \mathrm{tr}_{1,\dots,m} A^{(m)} \mu_{i_1} \dots \mu_{i_k} (-\partial_z + E_{j_1} z^{-1}) \dots (-\partial_z + E_{j_{m-k}} z^{-1}) 1,$$

summed over disjoint subsets of indices $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{m-k}\}$ of $\{1, \dots, m\}$. Therefore,

$$\phi_m^{(k)} = z^{m-k} \binom{m}{k} \mathrm{tr}_{1,\dots,m} A^{(m)} \mu_1 \dots \mu_k (-\partial_z + E_{k+1} z^{-1}) \dots (-\partial_z + E_m z^{-1}) 1.$$

By calculating the partial trace of the anti-symmetrizer with the use of (4.4), we get

$$\phi_m^{(k)} = \binom{m}{k} \mathrm{tr}_{1,\dots,m} A^{(m)} \mu_1 \dots \mu_k E_{k+1} \dots E_m + \sum_{r=k+1}^{m-1} c_r \mathrm{tr}_{1,\dots,r} A^{(r)} \mu_1 \dots \mu_k E_{k+1} \dots E_r$$

for certain constants c_r . The same argument applied to the expansion defining the elements $\varphi_m^{(k)}$ gives

$$\varphi_m^{(k)} = \binom{m}{k} \mathrm{tr}_{1,\dots,m} A^{(m)} \mu_1 \dots \mu_k E_{k+1} \dots E_m.$$

This yields a triangular system of linear relations

$$\phi_m^{(k)} = \varphi_m^{(k)} + \sum_{r=k+1}^{m-1} c_r \varphi_r^{(k)}.$$

Since $\phi_{k+1}^{(k)} = \varphi_{k+1}^{(k)}$, we can conclude that the elements $\varphi_m^{(k)}$ are generators of \mathcal{A}_μ . The argument for the elements $\psi_m^{(k)}$ is quite similar. Taking into account the properties of the elements $\overline{\varphi}_m^{(k)}$ and $\overline{\psi}_m^{(k)}$, we come to another corollary.

Corollary 4.6. *The elements of each of the two families $\varphi_m^{(k)}$ and $\psi_m^{(k)}$ associated with the boxes of the skew diagram Γ/γ as in (1.9) are algebraically independent generators of the algebra \mathcal{A}_μ . \square*

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